SELECTED PROBLEMS IN CLASSICAL FUNCTION THEORY

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Abstract. We discuss several problems in classical complex analysis that might appeal to graduate students and young researchers. Among them are possible extensions to multiply connected domains of the Neuwirth-Newman theorem regarding analytic functions with positive boundary values, characterizing domains by properties of best approximations of

$$\Big)^{\frac{1}{p}} =: \|f\|_p < \infty \Big\};$$

where $dA = \frac{1}{\pi}dxdy$ denotes normalized area measure in the unit disk $\mathbb{D}$. The Bergman spaces $A^p(G)$ for an arbitrary domain G are defined in a similar way.

If instead of area measure, we consider line integrals on concentric circles, we get the Hardy spaces.

Definition 1.2. For $0 < p < \infty$, define the Hardy space of the disk as

$$H^p(\mathbb{D}) := \Big\{ f \text{ analytic in } \mathbb{D} : \sup_{0<r<1} \frac{1}{2\pi}\int_0^{2\pi} |f(re^{it})|^p dt =: \|f\|^p_{H^p} < \infty \Big\}.$$

When $p = \infty$, we define

$$H^\infty(\mathbb{D}) = \{ f : f \text{ analytic in } \mathbb{D} : \sup\{|f(z)|; z \in \mathbb{D}\} =: \|f\|_\infty < \infty\}.$$

For arbitrary domains G, we define the Hardy spaces as follows (see [11]).

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we can replace non-tangential boundary values with asymptotic boundary values (see [19]), and interpret "almost everywhere" as being understood with respect to harmonic measure, and then the theorem extends to $H^p(G)$.

An interesting question is, what happens for the Smirnov classes $E^p(G)$ when $p \ge 1$? The answer depends on the geometric character of the boundary. Recall that a finitely connected domain G is called Smirnov if the derivative of the conformal map from a circular domain onto G is an outer function (see [8]). It turns out that if the domain G is non-Smirnov, then the theorem is false, and indeed in such a domain, for every $0 < p < \infty$, there exist non-constant $f \in E^p(G)$ such that $0 \le f(\zeta) \le 1$ for almost every ζ on the boundary of the domain (see [18]). In the other extreme, if the domain G has smooth boundary Γ, then the classes $H^p(G)$ and $E^p(G)$ are equal (as sets), and therefore the situation in $E^p(G)$ is exactly the same as that of $H^p(G)$. However, if the domain is Smirnov and has singularities, these singularities allow for the construction of functions in E^p with real boundary values, for certain values of p. In fact, the values of p that allow for the construction of such functions are tightly connected in general to the geometric characteristic of the singularity. For more detail on the construction of such functions with real boundary values, see [6, 7] and the references therein.

In the case that there do exist non-trivial functions in E^p with real boundary values, if G is a simply connected Smirnov domain, L. DeCastro and D. Khavinson noted that the analogue of the Neuwirth-Newman theorem holds:

Theorem 2.2. ([7]) *Let G be a simply connected Smirnov domain with rectifiable boundary Γ. Let $p_0 \ge 1$ be defined as the smallest $p \ge 1$ such that $f \in E^p(G)$ and f has real boundary values a.e. on Γ imply that f is a constant. Then all $f \in E^{p_0/2}$ such that $f \ge 0$ a.e. on Γ are constants.*

The proof of this theorem is along the same lines as that of the original Neuwirth-Newman result, and is sketched here.

Proof. Write $f(z) = B(z)S(z)F^2(z)$, where $B(z)$ is a generalized Blaschke product, $S(z)$ is a bounded singular inner function, and $F(z) \in E^{p_0}$ is an outer function. On Γ, since $f \ge 0$, we have that $B(z)S(z)F^2(z) = |f(z)|$. On the other hand, $|f(z)| = |F(z)|^2 = F(z)\overline{F(z)}$ a.e., and therefore $\overline{F(z)} = B(z)S(z)F(z) \in E^{p_0}(G)$. This implies that $F(z) + \overline{F(z)} \in E^{p_0}(G)$ and is real-valued, hence a constant. Thus, $f(z) = \text{const} \cdot B(z)S(z)$ is a bounded function with non-negative boundary values, hence (

Theorem 3.2. ([21]) Let G be a nitely connected region whose boundary is analytic, and let A and P be the area and perimeter of G. The following are equivalent:

(i) = $\frac{2A}{P}$;

(ii) There is 2 H^1 (G) such that z(s) i dz

Sketch of the proof: Without loss of generality, let's assume that the best approximation is zero. Then one can show that $0 \in G$ and that the extremal function f^* for the dual problem satisfies $|f^*| \le 1$ in G and $|f^*| = 1$ on the boundary, and the duality relationship

$$f^* z\,dz = \text{const}\,|z|^p ds \quad \text{on } \Gamma$$

holds, where we can take the constant to be positive. Dividing by z^2 yields

$$\frac{f^*(z)}{z}dz = \text{const}\,|z|^{p-2}ds. \tag{3.1}$$

For $p = 1$, using regularity results for extremals (see [23]) in order to apply the argument principle, if f^* is not constant, one can show that the left hand side of (3.1) has a non-trivial increment of its argument, while the right hand side doesn't (because it's positive), which is a contradiction. Therefore, we conclude that f^* is a unimodular constant. Now again using regularity of the boundary and parametrizing $z = r(\theta)e^{i\theta}$, and using the duality relationship (3.1) gives, after some simple calculus, that $dr/d\theta = 0$; and hence Γ consists of circles centered at the origin. Using the duality equation one last time shows that since dz/ds must have the same sign on both circles, then there can only be one circle, and hence, G is a disk. The case $p > 1$ is more complicated, and in particular, the case that $p \in \mathbb{N}$ has to be treated separately. For details, see [15].

Note that this theorem *proves* that the domain is simply connected. If we assume G to be simply connected to begin with, the regularity hypothesis (that is, the analyticity of the boundary) can be relaxed significantly to assume merely that G is a Smirnov domain, by appealing to the following theorem.

Theorem 3.5. ([10]) *Let G be a Jordan domain in $\mathbb{C}$ containing 0 and with the rectifiable boundary Γ satisfying the Smirnov condition. Suppose the harmonic measure on Γ with respect to the origin equals $c|z|^{\alpha} ds$ for $z \in \Gamma$, where ds denotes arclength measure on Γ, $\alpha \in \mathbb{R}$ and c is a positive constant. Then*

(i) For $\alpha = -2$, the solutions are precisely all disks G containing 0.

(ii) For $\alpha = -3, -4, -5, \ldots$ there are solutions G which are not disks.

(iii) For all other values of α, the only solutions are disks centered at 0.

The conclusion of Theorem 3.4 then follows, because the left hand side of (3.1) is a constant multiple of harmonic measure at the origin, and since in our case, $\alpha = p - 2$ with $p > 1$ so $\alpha > -1$, part (iii) of Theorem 3.5 applies, giving that G is a disk.

What happens in the finitely connected case is not known, and thus leads to the following problem.

Problem 3.1. *Extend Theorem 3.4 to finitely connected Smirnov domains. In particular, do the hypotheses of that theorem imply that G is simply connected?*

Notice that in the case $0 < p < 1$, we can still define analytic content, but we lose duality (since in that case E^p is not a Banach space), and so it is not clear

where u is a solution of the dual problem. Integrating with respect to z gives (in G):

$$u = \text{const}|z|^p + h; \text{ where } h \in H^1(G):$$

Since $u = 0$ on ; we get that $h|$ is real-valued and therefore constant on; and therefore $|z|$ is constant on ; hence is a disk. For the proof of (ii), see [15].

Note that this proof is easier in the context of Bergman spaces, because the duality relationship holds in the whole domain G.

Problem 3.4. What are the isoperimetric "sandwich" estimates for A^p?

Nothing is known about the following problem.

Problem 3.5. What can be said about domains with other rational best approximations of z in A^p? For example, if the best approximation is a rational function of degree 2, what is the corresponding domain?

4. Putnam's Inequality for Toeplitz Operators in Bergman Spaces

This yields an alternative proof that Saint-Venant's inequality becomes equality only for disks. We are thus left with a host of interesting problems to investigate.

Problem 4.1. Find the "book" proof of the Olsen - Reguera theorem in [25], freeing it from the power series calculation and extending the result to arbitrary domains.

Problem 4.2. Is the sharp upper bound for the A^2-content equal to $\frac{1}{2}\sqrt{\frac{\text{Area}(G)}{\pi}}$?

Problem 4.3. What is the sharp lower bound for the A^2-content expressed in terms of geometric characteristics (e.g., area, perimeter, principal frequency) of the domain?

Problem 4.4. Refine the "isoperimetric sandwich" inequalities for $\|[T^*, T]\|$ to include the connectivity of the domain.

This last problem is virtually unexplored territory. In his thesis in the 70s ([17]), S. Jacobs refined Carleman's celebrated inequality ([5]) bounding the A^2 norm of G in terms of the E^1 norm of G for multiply connected domains. In [22], there is a result connecting geometric characteristics of the domain G (area, perimeter, connectivity, and analytic content) with the mapping properties of φ, the best approximation of $\bar{z}$, and the mapping properties of the extremal function in the dual problem.

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