THE ISOPERIMETRIC INEQUALITY VIA APPROXIMATION THEORY AND FREE BOUNDARY PROBLEMS

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Abstract. In this survey paper, we examine the isoperimetric inequality

One of the early analytic proofs of (1.1) was given by A. Hurwitz [26] in 1901. Let us sketch his argument here.

Sketch of proof. Suppose the region Ω is bounded by the simple closed smooth curve $\Gamma,$ parametrized with respect to the arc-length parameter s and with length $2\pi.$ (So, the isoperimetric inequality would state that $A \leq \pi.$

problems. This approach reveals a close tie to hydrodynamics and, in particular, to problems concerning shapes of electrified droplets of perfectly conducting fluid. We use as a point of departure the paper [32], in which the author discusses the concept of analytic content, and the related survey paper [18]. In Section 3, we discuss the connection with overdetermined boundary value problems and Serrin's theorem. In Section 4, we describe a more general problem and its application to determining the shape of a droplet of conducting fluid in the presence

This theorem is discussed in detail in [18]. Let us outline the argument here. H. Alexander in 1973 ([3]) proved the upper estimate by noticing the connection with the Ahlfors-Beurling estimate from 1950 ([1]).

More specifically, suppose D is a bounded domain (with smooth boundary ∂D) containing $\bar{\Omega}$. By the Cauchy-Green formula,

$$\bar{\zeta} = \frac{1}{2\pi i}\int_{\partial D}\frac{\bar{z}}{z-\zeta}\,dz - \frac{1}{\pi}\int_{D}\frac{1}{z-\zeta}\,dA(z),$$

where dA is area measure. Define

$$G(\zeta) = \frac{1}{\pi}\int_{\Omega}\frac{1}{z-\zeta}\,dA(z).$$

Then

$$\bar{\zeta} + G(\zeta) = \frac{1}{2\pi i}\int_{\partial D}\frac{\bar{z}}{z-\zeta}\,dz - 1$$

Theorem 2.2. *([32]) Let Ω and Γ be as above. The following are equivalent:*

(i) $\lambda = \frac{2A}{P}$;

(ii) There is $\varphi \in A_\Omega$ *such that* $\bar{z}(s) - i\lambda \dot{\bar{z}}(s) = \varphi(z(s))$ *on* Γ*; where* s *is the arc-length parameter;*

(iii) $\frac{1}{A}\int_\Omega f\,dA = \frac{1}{P}\int_\Gamma f\,ds$ *for all* $f \in A_\Omega$*:*

Remark. Note that (iii) holds for annuli $\Omega = \{r < |z| < R\}$. Simply take the Laurent series decomposition of $f = f_1 + f_2$ in the annulus, where f_1 is analytic inside $\{z : |z| < R\}$ and $f_2(\infty) = 0$; and notice that both sides of the equality in (§

The left hand side of equation (2.3) is real and therefore has argument increment 0, while the right hand side has an argument increment of at least 4π as we travel along Γ, unless $\varphi' = 0$. (Note that we need to be careful if the expression on the left hand side passes through a zero on $\partial\Omega$.) Hence, $\varphi = const$ and (2.1) implies that $\Gamma = \{z : |z - const| = \lambda\}$, a disk. □

Equation (2.1) is closely connected to the "Riccati equation": since Γ is analytic, near each component γ_j there is a single-valued branch of an analytic function $S(z)$ (the Schwarz function, see [2, 12, 57] and also Section 5) such that $\bar{z} = S(z)$ on Γ. Then

$$u := \sqrt{S'(z)}$$

is a single-valued analytic function in a tubular neighborhood of $\partial\Omega$, and

$$u(z) = \frac{d\bar{z}}{ds} \text{ on } \Gamma,$$

so, after differentiating one more time with respect to the arc-length parameter, (2.1) becomes the Riccati equation

$$u^2 - i\lambda u' = f, \tag{2.4}$$

where $f = \varphi'$. Since Riccati's equation is easily transformed into a homogeneous second order linear equation (see [27]) which may only have two linearly independent solutions, it is yet another indication that if (2.1) holds on Γ, Ω must be at most doubly connected and disks and annuli are the only domains for which (2.1) may hold. Yet Conjecture 2.1 is still open even for doubly connected domains! The Riccati equation (2.4) appears in many free boundary problems, some of which we will discuss in more detail in Section 4. We now turn to the connection with overdetermined boundary problems and Serrin's theorem.

3. Overdetermined Boundary Value Problems and Serrin's Theorem

Recall that condition (iii) from Theorem 2.2 states:

$$\frac{1}{A}\int_\Omega f\,dA = \frac{1}{P}\int_\Gamma f\,ds$$

for analytic f in Ω. If Ω is simply connected, this condition is equivalent to:

$$\frac{1}{A}\int_\Omega u\,dA = \frac{1}{P}\int_\Gamma u\,ds \tag{3.1}$$

for all functions u harmonic in Ω. Moreover, A. Kosmodem'yansky showed that condition (3.1) is equivalent to the following.

Theorem 3.1. *([39]) Consider the solution v of the Dirichlet problem*

$$\Delta v = 1 \text{ in } \Omega;\ v = 0 \text{ on } \Gamma.$$

Then the normal derivative of v must satisfy $v_n = A/P$ on Γ.

Indeed, take any harmonic test function u in Ω that is smooth up to the boundary. By Green's formula,

$$\int_{\Gamma} u v_n ds = \int_{\Omega} u dA = \frac{A}{P} \int_{\Gamma} u ds.$$

Since u is arbitrary, $v_n = A/P$ on Γ. In this context, the shape of Ω was already known. We state the following result due to Serrin in two dimensions, although the theorem is more general and holds in all dimensions.

Theorem 3.2. ([56]) *If the overdetermined boundary value problem*

$$\Delta v = 1 \text{ in } \Omega,$$

$$v = 0 \text{ on } \Gamma,$$

$$v_n = const \text{ on } \Gamma,$$

has a smooth solution in Ω, then Ω is a disk.

This leads to an equivalent form of Conjecture 2.1 "à la Serrin" ([35]):

Conjecture 3.1. *Let Ω be a multiply connected domain. If the overdetermined boundary value problem ($n \geq 2$)*

$$\Delta v = 1 \text{ in } \Omega$$

$$\frac{\partial v}{\partial n} = A$$

where p is the pressure at each point (x, y, z) and depends only on z, since the flow is laminary. Since $\frac{\partial^2 p}{dz^2} = 0$, the right hand side is actually a constant A, giving us (modulo a constant multiple) the first equation in Serrin's theorem.

On the other hand, the force exerted by the water on the walls of the pipe is given by

$$\vec{F} = (p - \frac{4}{3}\mu \nabla \cdot \vec{v})$$

We now turn to a discussion of condition (ii) in Theorem 2.2 and a related application to determining the shape of droplets of conducting fluid in the presence of an electric field.

4. Droplets

Recall one of the equivalent conditions for $\lambda(\Omega) = 2A/P$:

$$\ddot{z}(s) - i\lambda\dot{z}(s) = \varphi(z(s))$$

for some $\varphi \in A_\Omega$. We would like to consider a more general problem, in which the function φ may not be continuous in the closure of Ω and may possibly have exists by Koebe's theorem (see [20, p. 237-238]). Since Ω has a rectifiable boundary Γ, φ' can be shown to be in $E^1(\Omega)$. We say that Ω is a *Smirnov* domain if, for each $z \in \Omega$,

$$\log|\varphi'(z)| = \frac{1}{2\pi} \int_{\Gamma} \log|\varphi'(\zeta)| \frac{\partial g_{\Omega}(}{}$$

Theorem 4.2. *(*[36, p. 24-26]*) There exists a one parameter family of unbounded domains Ω_t, each with rectifiable boundary Γ_t, and a corresponding family of functions F_t analytic in Ω_t except for a simple pole with residue 1 at ∞, such that*

$$F_t(z) = p_t\bar{z} + i\tau_t\frac{d\bar{z}}{ds} \text{ on } \Gamma_t,$$

for some real constants p_t and τ_t, with $p_t \neq 0$, $\tau_t \neq 0$.

Each of these domains Ω_t is thus an example of a solution to Problem 4.1. Their boundaries Γ_t are images of the unit circle under a rational mapping of degree 3 on which (4.1) holds. None of these curves however is a physical droplet. To our knowledge, no other examples of such domains are known. In particular, we do not know of any examples of transcendental curves satisfying (4.1), although, most likely, there are plenty of them!

Applying electrical forces to droplets of conducting fluid has led to some very concrete applications: the process of "electrowetting", for example, in which an electric force is applied at the interface of a droplet of conducting fluid and a solid, has applications to digital cameras, camera phones, and home security systems. In 2003, scientists from Philips Research created a fluid lens that operates on the basis of the process of electrowetting: two non-mixing fluids, one conducting and one not, are placed inside a tube. The layer between the liquids (the *meniscus*) acts as a lens. An electric field is applied to the tube, which causes the conducting fluid to change its shape, thus resulting in a change of the focal length of the lens. See [49] for more details. For further references on electrowetting and its applications, see [5, 24]. A slightly different type of application can be found in [11]: there, the authors use Schwarz functions to model the changing shape of a void created and traveling inside a thin metal conductor subjected to an intense electric field. This model is similar in some ways to the one used for Hele-Shaw flows (see [10, 23, 53]).

5. Some special cases

Let us now examine three distinguished cases of Problem 4.1, in which the boundary condition on $\Gamma = \cup_{j=1}^{n}\gamma_j$ simplifies to one of the following:

(5.1) $$F(z) = p_j\bar{z} \qquad z \in \gamma_j,\ p_j \in \mathbb{R} - \{0\},$$

(5.2) $$F(z) = i\tau_j\frac{1}{\bar{z}} \qquad z \in \gamma_j,\ \tau_j \in \mathbb{R} - \{0\},$$

(5.3) $$F(z) = p_j\bar{z} + c_j \qquad z \in \gamma_j,\ p_j \in \mathbb{R} - \{0\}, c_j \in \mathbb{C}.$$

Note that the existence of a function F satisfying (5.2) implies the existence of a function g satisfying (5.3): simply define

$$g(z) = \int (F(z))^2 dz.$$

Then, by (5.2), for $z \in \gamma_j$, we have

$$\int (F(z))^2 dz = -\tau_j^2 \int \left(\frac{d\bar{z}}{ds}\right)^2 dz = -\tau_j^2(\bar{z} + c_j),$$

for some constant c_j. Therefore g is well-defined as a single valued analytic function, and (5.3) holds. From now on, we shall always assume additional regularity for Ω, i.e., that Ω is a Jordan Smirnov domain.

5.1. The Schwarz function.

generality $p_j = 1$), then Ω is a so-called *quadrature domain*; namely, if f is any function analytic in Ω, then by the complex form of Green's theorem,

$$\int_{\Omega} f\,dA = \frac{1}{2i}\int_{\Gamma} f\bar{z}\,dz = \frac{1}{2i}\int_{\Gamma} fF\,dz = \pi\sum_{j=1}^{m} f(a_j)\operatorname{Res}_{a_j} F.$$

These domains have been intensely studied in the 1980s by D. Aharonov, B. Gustafsson, H. S. Shapiro, K. Ullemar, Y. Avsi (see [57] and references therein). Also, see [23] for an account of many recent developments.

Even when we do not require the coefficients p_j to be equal, a similar argument as in the proof of Theorem 5.1 shows that if F is assumed to have a simple pole at the origin and Ω is bounded, then Ω must be a disk. (This is well-known in the context of Schwarz functions: the Schwarz function of a domain has one pole if and only if the domain is a disk.) If the function F has two different poles (and if the coefficients p_j are different), then the problem is already more difficult.

5.2. **Vekua's Problem.** The second special case (5.2)

$$F(z) = i\tau_j \frac{1}{z} \qquad z \in \gamma_j;\ \tau_j \in \mathbb{R} \setminus \{0\}$$

is a particular example of an overdetermined boundary value problem made enormously popular by works of I. N. Vekua in the 1950s. It is not difficult to

$z) =$

holds in all dimensions provided that Γ is a C^2 surface (see [52]). Equivalently, if the overdetermined boundary value problem

$$\Delta u = 0 \ in \ \Omega;$$
$$u = const \neq 0 \ on \ \Gamma;$$
$$\frac{\partial u}{\partial n} = const \ on \ \Gamma$$

has a solution in an (unbounded!) domain Ω, then Γ is a circle.

Remark. In [14], the authors notice that it is possible to drop the assumption that the domain is Smirnov, but then instead one must assume that the function F is in E^2; since the proof uses the fact that the function $z^2(F(\varphi(z)))^2\varphi'(z)$ is in $H^1(\mathbb{D})$ (where φ is the Riemann mapping from the disk to Ω), and therefore cannot coincide with the conjugate of an H^1 function on the circle. It is not clear whether the theorem itself fails if one drops the assumption that Ω is Smirnov and considers only $F \in E^1$. In this context, one must cautiously observe that in non-Smirnov domains, there exist functions with positive and bounded boundary values which belong to any E^p class, $p < \infty$ (see [30]).

We may also consider the case where F has a simple pole at infinity. Recall that this context has a physical interpretation, discussed in Section 4, as a droplet of conducting fluid in which the surface tension is much larger than the pressure inside the droplet (which is then considered negligible). In this case, the following theorem gives an example of a family of mathematical droplets.

Theorem 5.4. *(*[36, Thm 6.2]*) Let Γ be a Jordan curve, with (logarithmic) capacity* 1*, whose exterior Ω is a Smirnov domain. If $\tau \geq \frac{3+2\sqrt{3}}{3}$ and there exists $F \in E^1$ near the boundary of Ω and with a simple pole at ∞, that is, $F = z + O(\frac{1}{z})$, and*

$$F = i\tau\frac{d\bar{z}}{ds} \ on \ \Gamma, \tag{5.5}$$

then Γ is included into one parameter family $\{\Gamma_t\}$, $t = 1/\tau$, where Γ_t is the image of the unit circle under the conformal mapping

$$\varphi_t(w) = \frac{1}{w} - 2tw - \frac{t^2}{3}w^3.$$

For $\tau \leq \frac{3+2\sqrt{3}}{3}$, (5.5) *has no solution among mathematical droplets with Jordan boundaries. The droplets are convex for $\tau \geq 3$ and the family contains only one physical droplet corresponding to the value $\tau = 3$.*

6. Extensions to higher dimensions

Finally, let us discuss what is known in higher dimensions. Suppose Ω is a bounded domain in $\mathbb{R}^n$, $n \geq 3$, Γ is the boundary of Ω, V is the volume of Ω, and P is the (surface) area of Γ. Let $H(\Omega)$ be the closure in the uniform norm on $\overline{\Omega}$ of the space of functions harmonic in a neighborhood of Ω. More generally, if K is a compact subset of $\mathbb{R}^n$, and $C(K)$ is the space of continuous functions on

K; we will write $H(K)$ for the uniform closure in $C(K)$ of the space of functions harmonic in a neighborhood of K.

Let $\mathbf{x} = (x_1, \ldots, x_n)$ be a vector in $\mathbb{R}^n$, and $|\mathbf{x}|^2 = (\sum_{j=1}^{n} x_j^2)^{\frac{1}{2}}$. If one thinks of $H(K)$ as the uniform closure of the kernel of the Laplace operator Δ and $R(K)$ as the uniform closure of the kernel of the operator $\partial/\partial\bar{z}$, then the analogy of the anti-analytic function $\bar{z}$ is the function $|\mathbf{x}|^2$, since $(\partial/\partial\bar{z})(\bar{z}) = 1$ and $\Delta(|\mathbf{x}|^2) = 2n = const \neq 0$. With this in mind, we define the concept of *harmonic content* as follows.

Definition. *The harmonic content of K is defined to be*

$$\Lambda(K) := dist_{C(K)}(|\mathbf{x}|^2, H(K)).$$

For a bounded domain Ω, we will write $\Lambda(\Omega) := \Lambda(\overline{\Omega})$. We then have the following result.

Theorem 6.1. *(*[33]*)*

$$\Lambda(K) = 0 \iff H(K) = C(K).$$

Note that in the case of analytic content in $\mathbb{C}$, the equivalence of the statements $\lambda(K) = 0$ and $R(K) = C(K)$ follows at once from the Stone-Weierstrass theorem, since $R(K)$ is an algebra. However, Theorem 6.1 is non-trivial, since $H(K)$ is not an algebra. Different proofs were given by Poletsky ([50]) and Bliedtner (see [7] and references therein, in particular to the works of W. Hansen).

Harmonic content can be estimated in terms of geometric quantities. If R_{harm} is the radius of the ball with the same capacity as $\overline{\Omega}$, and R_{vol} is the radius of the ball with the same volume as Ω, then the following theorem gives upper and lower bounds for the harmonic content of a domain Ω.

Theorem 6.2. *(*[33, 34]*)*

$$\frac{1}{2}R_{harm}^2 \leq \Lambda(\Omega) \leq \frac{1}{2}R_{vol}^2$$

and equality on either side occurs only for balls.

The upper estimate was proved in [33], and the lower estimate as well as extensions of both inequalities to general elliptic operators were obtained in [34]. An interesting extension of this result to approximation in C^1-norm by harmonic

Recall that analytic content for a domain Ω in $\mathbb{C}$ is defined as

$$\lambda(\Omega) := \inf_{\varphi \in A(\Omega)} \|\bar{z} - \varphi\|_{C(\partial\Omega)}.$$

Note that this is also equal to

$$\lambda(\Omega) := \inf_{\varphi \in A(\Omega)} \|z - \psi\|_{C(\partial\Omega)}.$$

An anti-analytic function $\psi = f_1 + i f_2$ can be identified with the harmonic vector field $\vec{f} = (f_1, f_2) = \nabla u$, u a harmonic real-valued function, where

$$\operatorname{Div} \vec{f} = \operatorname{Curl} \vec{f} = 0.$$

the extremal solids are *not* balls in all dimensions ≥ 3; although they are very symmetric algebraic surfaces that are getting more and more tightly sealed to the tangent plane at the maximum point (see [22, p. 82]). The following conjecture proposed in [22] remains open.

Conjecture 6.1. $\lambda(\Omega) \leq R_{vol}$.

The following theorem is the analogue of Theorem 2.2 and gives conditions equivalent to the attainment of the lower bound in Theorem 6.3.

Theorem 6.4. *([22]) TFAE:*

(i) $\lambda(\Omega) = \frac{nV}{P}$;

(ii) There exists $\vec{\beta} \in B(\Omega)(\omega)$: $\vec{x} - \lambda \vec{n}(x) = \vec{\beta}(x)$ *on* $\partial\Omega$; *where* $\vec{n}$ *is the outward unit normal to* $\partial\Omega$.

(iii) $\frac{1}{V}\int_{\Omega} u\,dV = \frac{1}{P}\int_{\partial\Omega} u\,d\sigma$ *for all u harmonic in* Ω *such that* $\int_S \frac{\partial u}{\partial n} d\sigma = 0$ *for all closed surfaces S in* Ω.

(iv) There exists u in Ω *satisfying*

$$\Delta u = 1;$$
$$\frac{\partial u}{\partial n}\Big|_{\partial\Omega} = const;$$
$$u|_{\partial\Omega} = local\ constant.$$

The following conjecture thus follows naturally:

Conjecture 6.2. $\lambda(\Omega) = \frac{nV}{P} \iff \Omega$ *is either a ball or a spherical shell.*

Serrin's theorem in higher dimensions implies that if an extremal domain is homeomorphic to a ball, then it must be a ball; however Conjecture 6.2 is still open for domains whose boundary contains more than one component, or domains (such as a torus) that are not homeomorphic to a ball.

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