

Extremal Problems for Nonvanishing Functions in Bergman Spaces

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Dedicated to the memory of Semeon Yakovlevich Khavinson.

Abstract. In this paper, we study general extremal problems for non-vanishing functions in Bergman spaces. We show the existence and uniqueness of solutions to a wide class of such problems. In addition, we prove certain regularity results: the extremal functions in the problems considered must be in a Hardy space, and in fact must be bounded. We conjecture what the exact form of the extremal function is. Finally, we discuss the specific problem of minimizing the norm of non-vanishing Bergman functions whose first two Taylor coefficients are given.

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1. Introduction

Let $1 < p < \infty$,
 $A^p = \{f \in L^p(\mathbb{D}) : \int_{\mathbb{D}} |f(z)|^p dA(z) < \infty\}$
 where $dA = \frac{1}{\pi} dx dy$ is the Lebesgue measure on \mathbb{D} , $z = x + iy$, $1 \leq p < \infty$, A^p
 $H^p = \{f \in L^p(\mathbb{D}) : \int_0^1 \int_0^{2\pi} |f(re^{i\theta})|^p \frac{d\theta}{2\pi} dr < \infty\}$.

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$L^p(dA)$ (for $p \geq 1$) is a Banach space. The norm is defined by

$$\|f\|_p = \left(\int |f|^p dA \right)^{1/p}$$
 for $f \in L^p(dA)$. For $p = \infty$, the norm is defined by

$$\|f\|_\infty = \text{ess sup } |f|$$
 for $f \in L^\infty(dA)$. The space $L^\infty(dA)$ is also a Banach space.

$$\left\{ \int w f dA \mid f \in A^p, \|f\|_p \leq 1 \right\}, \quad (6.6)$$

where $w \in L^q(dA)$ and $1/p + 1/q = 1$. The set of all such integrals is a closed ball in \mathbb{R} .

$$\left\{ \int f dA \mid f \in A^p, \|f\|_p \leq 1 \right\}, \quad (6.7)$$

is a closed ball in \mathbb{R} . The set of all such integrals is a closed ball in \mathbb{R} .

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Handwritten musical score consisting of three staves. The notation includes various notes, rests, and dynamic markings. The first staff begins with a dynamic marking of *fz*. The second staff starts with a dynamic marking of *ff*. The third staff contains dynamic markings of *Ap* and *HP*. The score is written in a cursive, handwritten style.

$\cup_1 \dots \cup_n \dots \cup_{n+1} \dots \cup_{n+2} \dots \cup_{n+3} \dots \cup_{n+4} \dots \cup_{n+5} \dots \cup_{n+6} \dots \cup_{n+7} \dots \cup_{n+8} \dots \cup_{n+9} \dots \cup_{n+10} \dots$

Proof. (\dots)
 $0 \leq i \leq n$ f_k A^p $l_i(f_k)$ C_i

f^p
 A^p $p >$
 A^2
 p
 f
 $k(\cdot, j)$

$$k(z, w) = 1/(1 - wz)^2.$$

$f(z)$
 $(z$

30

... [A] [B] [C] [D] [E] [F] [G] [H] [I] [J] [K] [L] [M] [N] [O] [P] [Q] [R] [S] [T] [U] [V] [W] [X] [Y] [Z] [aa] [ab] [ac] [ad] [ae] [af] [ag] [ah] [ai] [aj] [ak] [al] [am] [an] [ao] [ap] [aq] [ar] [as] [at] [au] [av] [aw] [ax] [ay] [az] [ba] [bb] [bc] [bd] [be] [bf] [bg] [bh] [bi] [bj] [bk] [bl] [bm] [bn] [bo] [bp] [bq] [br] [bs] [bt] [bu] [bv] [bw] [bx] [by] [bz] [ca] [cb] [cc] [cd] [ce] [cf] [cg] [ch] [ci] [cj] [ck] [cl] [cm] [cn] [co] [cp] [cq] [cr] [cs] [ct] [cu] [cv] [cw] [cx] [cy] [cz] [da] [db] [dc] [dd] [de] [df] [dg] [dh] [di] [dj] [dk] [dl] [dm] [dn] [do] [dp] [dq] [dr] [ds] [dt] [du] [dv] [dw] [dx] [dy] [dz] [ea] [eb] [ec] [ed] [ee] [ef] [eg] [eh] [ei] [ej] [ek] [el] [em] [en] [eo] [ep] [eq] [er] [es] [et] [eu] [ev] [ew] [ex] [ey] [ez] [fa] [fb] [fc] [fd] [fe] [ff] [fg] [fh] [fi] [fj] [fk] [fl] [fm] [fn] [fo] [fp] [fq] [fr] [fs] [ft] [fu] [fv] [fw] [fx] [fy] [fz] [ga] [gb] [gc] [gd] [ge] [gf] [gg] [gh] [gi] [gj] [gk] [gl] [gm] [gn] [go] [gp] [gq] [gr] [gs] [gt] [gu] [gv] [gw] [gx] [gy] [gz] [ha] [hb] [hc] [hd] [he] [hf] [hg] [hh] [hi] [hj] [hk] [hl] [hm] [hn] [ho] [hp] [hq] [hr] [hs] [ht] [hu] [hv] [hw] [hx] [hy] [hz] [ia] [ib] [ic] [id] [ie] [if] [ig] [ih] [ii] [ij] [ik] [il] [im] [in] [io] [ip] [iq] [ir] [is] [it] [iu] [iv] [iw] [ix] [iy] [iz] [ja] [jb] [jc] [jd] [je] [jf] [jg] [jh] [ji] [jj] [jk] [jl] [jm] [jn] [jo] [jp] [jq] [jr] [js] [jt] [ju] [jv] [jw] [jx] [jy] [jz] [ka] [kb] [kc] [kd] [ke] [kf] [kg] [kh] [ki] [kj] [kk] [kl] [km] [kn] [ko] [kp] [kq] [kr] [ks] [kt] [ku] [kv] [kw] [kx] [ky] [kz] [la] [lb] [lc] [ld] [le] [lf] [lg] [lh] [li] [lj] [lk] [ll] [lm] [ln] [lo] [lp] [lq] [lr] [ls] [lt] [lu] [lv] [lw] [lx] [ly] [lz] [ma] [mb] [mc] [md] [me] [mf] [mg] [mh] [mi] [mj] [mk] [ml] [mm] [mn] [mo] [mp] [mq] [mr] [ms] [mt] [mu] [mv] [mw] [mx] [my] [mz] [na] [nb] [nc] [nd] [ne] [nf] [ng] [nh] [ni] [nj] [nk] [nl] [nm] [nn] [no] [np] [nq] [nr] [ns] [nt] [nu] [nv] [nw] [nx] [ny] [nz] [oa] [ob] [oc] [od] [oe] [of] [og] [oh] [oi] [oj] [ok] [ol] [om] [on] [oo] [op] [oq] [or] [os] [ot] [ou] [ov] [ow] [ox] [oy] [oz] [pa] [pb] [pc] [pd] [pe] [pf] [pg] [ph] [pi] [pj] [pk] [pl] [pm] [pn] [po] [pp] [pq] [pr] [ps] [pt] [pu] [pv] [pw] [px] [py] [pz] [qa] [qb] [qc] [qd] [qe] [qf] [qg] [qh] [qi] [qj] [qk] [ql] [qm] [qn] [qo] [qp] [qq] [qr] [qs] [qt] [qu] [qv] [qw] [qx] [qy] [qz] [ra] [rb] [rc] [rd] [re] [rf] [rg] [rh] [ri] [rj] [rk] [rl] [rm] [rn] [ro] [rp] [rq] [rr] [rs] [rt] [ru] [rv] [rw] [rx] [ry] [rz] [sa] [sb] [sc] [sd] [se] [sf] [sg] [sh] [si] [sj] [sk] [sl] [sm] [sn] [so] [sp] [sq] [sr] [ss] [st] [su] [sv] [sw] [sx] [sy] [sz] [ta] [tb] [tc] [td] [te] [tf] [tg] [th] [ti] [tj] [tk] [tl] [tm] [tn] [to] [tp] [tq] [tr] [ts] [tt] [tu] [tv] [tw] [tx] [ty] [tz] [ua] [ub] [uc] [ud] [ue] [uf] [ug] [uh] [ui] [uj] [uk] [ul] [um] [un] [uo] [up] [uq] [ur] [us] [ut] [uu] [uv] [uw] [ux] [uy] [uz] [va] [vb] [vc] [vd] [ve] [vf] [vg] [vh] [vi] [vj] [vk] [vl] [vm] [vn] [vo] [vp] [vq] [vr] [vs] [vt] [vu] [vv] [vw] [vx] [vy] [vz] [wa] [wb] [wc] [wd] [we] [wf] [wg] [wh] [wi] [wj] [wk] [wl] [wm] [wn] [wo] [wp] [wq] [wr] [ws] [wt] [wu] [wv] [ww] [wx] [wy] [wz] [xa] [xb] [xc] [xd] [xe] [xf] [xg] [xh] [xi] [xj] [xk] [xl] [xm] [xn] [xo] [xp] [xq] [xr] [xs] [xt] [xu] [xv] [xw] [xx] [xy] [xz] [ya] [yb] [yc] [yd] [ye] [yf] [yg] [yh] [yi] [yj] [yk] [yl] [ym] [yn] [yo] [yp] [yq] [yr] [ys] [yt] [yu] [yv] [yw] [yx] [yy] [yz] [za] [zb] [zc] [zd] [ze] [zf] [zg] [zh] [zi] [zj] [zk] [zl] [zm] [zn] [zo] [zp] [zq] [zr] [zs] [zt] [zu] [zv] [zw] [zx] [zy] [zz]

... [math display="block">\int_{\mathbb{D}} |p_m^*(z)|^2 \operatorname{Re} \left(\prod_{i=1}^n (z - \alpha_i) \right) dA(z) \dots

$$F'(r) = \int_{\mathbb{D}} |p_m^*(z)|^2 \operatorname{Re} \left(\prod_{i=1}^n (z - \alpha_i) \right) dA(z) \dots$$

$$\int_{\mathbb{D}} |p_m^*(z)|^2 \operatorname{Re} \left(\prod_{i=1}^n (z - \alpha_i) \right) dA(z) \dots$$

$$\int_{\mathbb{D}} |p_m^*(z)|^2 \prod_{i=1}^n (z - \alpha_i) dA(z) \dots$$

... [math display="block">\int_{\mathbb{D}} |p_m^*(z)|^2 \prod_{i=1}^n (z - \alpha_i) dA(z) \dots ... \square

Lemma 2.7. For each $m \geq n$, $e^{\rho_m^*} \in H^2$, and the H^2 norm are bounded.

Proof.

$$p_m^*(z) = L(z) + h(z)q_{m-n}(z),$$

$$h(z) = \prod_{i=1}^n (z - \alpha_i), \quad q_{m-n}(z) = \prod_{i=1}^{m-n} (z - \beta_i),$$

$$\int_{\mathbb{D}} |e^{\rho_m^*(e^{i\theta})}|^2 d\theta = \int_{\mathbb{D}} |e^{\rho_m^*(z)}|^2 dA(z) \dots$$

$$\begin{aligned}
 & \dots \left[\frac{1}{z-k} \right] \dots \left[\frac{1}{z-k} \right] \dots \left[\frac{1}{z-k} \right] \dots \left[\frac{1}{z-k} \right] \dots \left[\frac{1}{z-k} \right] \dots \left[\frac{1}{z-k} \right] \dots \left[\frac{1}{z-k} \right] \dots \\
 & \dots q_{m-n-1} \dots \left[\frac{1}{z-k} \right] \dots \left[\frac{1}{z-k} \right] \dots \left[\frac{1}{z-k} \right] \dots \left[\frac{1}{z-k} \right] \dots \left[\frac{1}{z-k} \right] \dots \left[\frac{1}{z-k} \right] \dots \left[\frac{1}{z-k} \right] \dots
 \end{aligned}$$

$$zH'(z)q_{m-n}(z) = z \left\{ \sum_{k=1}^n \prod_{i=1, i \neq k}^n (z-i) \right\} \{ q_{m-n}(z-k) + (z-k)q_{m-n-1}(z) \}$$

$\psi_1 \dots \psi_n \in \mathbb{C}^n$ such that $\psi_1 \dots \psi_n = 0$

3. Another approach to regularity

Let $L = \begin{pmatrix} \lambda & & \\ & \lambda & \\ & & \lambda \end{pmatrix}$ and $\mathcal{D} = \begin{pmatrix} \lambda & & \\ & \lambda & \\ & & \lambda \end{pmatrix}$.
 Consider the system $L \psi = \mathcal{D} \psi$.
 This is equivalent to $(L - \mathcal{D}) \psi = 0$.
 The matrix $L - \mathcal{D}$ is $\begin{pmatrix} 0 & & \\ & 0 & \\ & & 0 \end{pmatrix}$.
 The system is satisfied for any $\psi \in \mathbb{C}^n$.
 The regularity condition is satisfied for any $\psi \in \mathbb{C}^n$.

$$\begin{aligned}
 & \left[\begin{array}{c} \mathbb{R} \\ \mathbb{R} \end{array} \right] \xrightarrow{f_s} \left[\begin{array}{c} \mathbb{R} \\ \mathbb{R} \end{array} \right] \\
 & \left[\begin{array}{c} \mathbb{R} \\ \mathbb{R} \end{array} \right] \xrightarrow{L(f/g_s)} \left[\begin{array}{c} \mathbb{R} \\ \mathbb{R} \end{array} \right] \xrightarrow{f_s} \left[\begin{array}{c} \mathbb{R} \\ \mathbb{R} \end{array} \right]
 \end{aligned}$$

• ... [, s , t , q , e : ' '] , ... [t , q , s ,] ...] , s

→ s , k [,] M ,
$$\int |f$$

$$f_{\lambda} \in \mathcal{C}(\mathbb{R}^n) \implies f_{\lambda} \in \mathcal{C}(\mathbb{R}^n) \quad \square$$

Proof. $f_{\lambda} \in \mathcal{C}(\mathbb{R}^n) \implies f_{\lambda} \in \mathcal{C}(\mathbb{R}^n) \implies f_{\lambda} \in \mathcal{C}(\mathbb{R}^n) \quad (*)$

... ..] , ...]

$$|a_s(z) - z| \leq |h_s(z) - \frac{1}{-z}| \leq \sum_{n=1}^{\infty} |b_{n,s} - 1| |z|^n. \quad (4.10)$$

... (4.10)

$$|b_{n,s} - 1| = \left| \frac{nt_0}{nt_0} - 1 \right|.$$

... ..

$$\frac{(x)/x - 1}{x^2}$$

... .. N

$$\left| \frac{nt_0}{nt_0} - 1 \right| \leq N(nt_0)^2 \leq N' n^2 s^2$$

... .. , fi] , ...

$$|a_s(z) - z| \leq N'' s^2 B(z),$$

...

$$B(z) = \sum_{n=1}^{\infty} n^2 |z|^n,$$

... ..

A musical score for piano and harp. The score is written on a grand staff with a treble clef and a bass clef. The piano part is on the upper staff, and the harp part is on the lower staff. The score includes various dynamic markings: A^2 , ff , and f . There are also performance instructions: A^p , H , and HP . The score is a single system of music.

Let $f(z) = \frac{1}{z} \prod_{j=1}^n (z - z_j)^2 S_j$, where $S_j = \int_{\gamma_j} g(z) dz$.

Let $S = \int_{\gamma} f(z) dz$, where γ is a contour enclosing all poles z_j . Then $S = \int_{\gamma} \frac{1}{z} \prod_{j=1}^n (z - z_j)^2 S_j dz$.

$$S = \int_{\gamma} \bar{f}^* \frac{1}{z} \left(\prod_{j=1}^n (z - z_j)^2 S_j \right) dz. \quad (1)$$

Let γ_j be a contour enclosing z_j . Then $S_j = \int_{\gamma_j} g(z) dz$.

$$S = \int_{\gamma} \bar{f}^* \frac{1}{z} \prod_{j=1}^n (z - z_j)^2 S_j dz = \int_{\gamma} \bar{f}^* \frac{1}{z} \prod_{j=1}^n (z - z_j)^2 g(z) dz$$

Let $K \subset \mathbb{T}_{f^*}$ be a compact set in A^2 containing the points $1, 2, \dots, n$. Then

$$f^* = \sum_{j=1}^n \frac{a_j}{(1-jz)^2}$$

is a meromorphic function in A^2 with poles at $1, 2, \dots, n$.

Corollary 4.2. If f^* is cyclic in A^2 , then m must be a rational function of the form (4.1).

Let $f^* = \sum_{j=1}^n \frac{a_j}{(1-jz)^2}$ be a cyclic function in A^2 . Then S is a subset of \mathbb{C}^n defined by

$$H = \left\{ \rho \in \mathbb{D} \mid \sum_{j=1}^n \frac{a_j}{(1-j\rho)^2} = 0 \right\}.$$

Since f^* is cyclic, H is a set of measure zero in \mathbb{D} . Let S be the set of points $(a_1, \dots, a_n) \in \mathbb{C}^n$ such that $\sum_{j=1}^n \frac{a_j}{(1-j\rho)^2} = 0$ for some $\rho \in \mathbb{D}$. Then S is a subset of H^2 .

$$B_r = \{f \in f_{A^2} \mid r \leq |f| \leq r\}$$

(z)

Let $\{e^j\}_{j=1}^n$ be the standard basis of \mathbb{C}^n .

Let f^j be the j -th row of $S(z)$.

$$\Lambda(z) = (S(z)(e^j))_{j=1}^n.$$

Let

$$S(z) = \frac{1}{\lambda} \int_0^1 \frac{e^j + z}{e^j - z} d(\cdot) \quad (6.4)$$

where $\lambda = \int_0^1 d(\cdot)$ and $\Lambda(\Sigma_r)$ is the r -th order approximation of $\Lambda(z)$.

$$\begin{aligned}
 & \text{...} \\
 & E \subset \mathbb{T}, \quad R(e_{f^2}^i) \text{ ...} \\
 & \text{...} \\
 & +\infty \text{ ...} \\
 & \text{...}
 \end{aligned}$$

\dots

(\dots) , ff , $R >$, ff , (\dots) , R , n , $n-1$, $R'(z)$, \mathbb{C}

Let f be a function in $H^2(\mathbb{D})$. Then f can be written as $f(z) = \sum_{n=0}^{\infty} a_n z^n$ for some sequence $\{a_n\}_{n=0}^{\infty}$ of complex numbers. The norm of f is given by $\|f\|_2^2 = \sum_{n=0}^{\infty} |a_n|^2$.

$$\int_{\mathbb{D}} |F'(z)|^2 dA = F(r) \cdot F'(r) \cdot F''(r) \cdot b \cdot F'(z) / r \in \mathbb{D}. \quad (5.1)$$

Let f be a function in $H^2(\mathbb{D})$. Then f can be written as $f(z) = \sum_{n=0}^{\infty} a_n z^n$ for some sequence $\{a_n\}_{n=0}^{\infty}$ of complex numbers. The norm of f is given by $\|f\|_2^2 = \sum_{n=0}^{\infty} |a_n|^2$.

$$\int_{\mathbb{D}} |f|^2 dA = f / r \in \mathbb{D}, f(r) \cdot f'(r) \in C. \quad (5.2)$$

Let f be a function in $H^2(\mathbb{D})$. Then f can be written as $f(z) = \sum_{n=0}^{\infty} a_n z^n$ for some sequence $\{a_n\}_{n=0}^{\infty}$ of complex numbers. The norm of f is given by $\|f\|_2^2 = \sum_{n=0}^{\infty} |a_n|^2$.

$$f(z) = C(z-A)e^{\mu_0 \frac{z+1}{z-1}}, \quad (5.3)$$

where $\mu_0 \geq 0$, $C, A \in \mathbb{C}$, $|\mu_0| \leq 1$.

$$f^* = c + cz^2$$

Let f^* be a function in $H^2(\mathbb{D})$. Then f^* can be written as $f^*(z) = \sum_{n=0}^{\infty} a_n z^n$ for some sequence $\{a_n\}_{n=0}^{\infty}$ of complex numbers. The norm of f^* is given by $\|f^*\|_2^2 = \sum_{n=0}^{\infty} |a_n|^2$.

$$f^* = hS,$$

where h is a function in $H^2(\mathbb{D})$ and S is a function in $H^2(\mathbb{D})$.

$$\int_{\mathbb{D}} |f^*|^2 z^{n+2} dA = n, n = 0, 1, \dots \quad (5.4)$$

Let f^* be a function in $H^2(\mathbb{D})$. Then f^* can be written as $f^*(z) = \sum_{n=0}^{\infty} a_n z^n$ for some sequence $\{a_n\}_{n=0}^{\infty}$ of complex numbers. The norm of f^* is given by $\|f^*\|_2^2 = \sum_{n=0}^{\infty} |a_n|^2$.

Proposition 5.1. If the inner factor S of f^* has a finite angular measure $d\mu$ then it is a finite Blaschke product, then

$$f^*(z) = C(z - \mu_0)e^{\mu_0 \frac{z+1}{z-1}} \quad (5.5)$$

where C and the real number μ_0 are uniquely determined by the interpolating condition.

Remark.

11

... 1000 ...

$$R(e^i) = \frac{(e^i - a)(\bar{a} - \bar{e}^i)}{e^i - \bar{a}e^i}, \quad |e^i - a|^2, \quad (1)$$

$$\int \{ \int R(e^i) d\mu(\cdot) \mid \mu \leq \nu, \mu \perp d \} \quad (2)$$

$$\int |h|^2 |S_\mu|^2 dA \leq \dots \quad (3)$$

1. 2. 3. 4. 5. 6. 7. 8. 9. 10.

$$\int_{\gamma} \frac{f(z)}{z} dz = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{z} dz = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{z} dz = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{z} dz$$

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10

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