Operator Theory:
Advances and Applications, Vol. 158, 59–86

Extremal Problems for Nonvanishing Functions in Bergman Spaces

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Dedicated to the memory of Semeon Yakovlevich Khavinson.

Abstract. In this paper, we study general extremal problems for non-vanishing functions in Bergman spaces. We show the existence and uniqueness of solutions to a wide class of such problems. In addition, we prove certain regularity results: the extremal functions in the problems considered must be in a Hardy space, and in fact must be bounded. We conjecture what the exact form of the extremal function is. Finally, we discuss the specific problem of minimizing the norm of non-vanishing Bergman functions whose first two Taylor coe cients are given.

Mathematics Subject Classification (2000). Primary: 30A38, Secondary: 30A98.

1. Introduction

$< p < \infty,$

$$A^p \quad \{f \quad \mathbb{D} \quad (\int |f(z)|^p dA(z))^{\frac{1}{p}} \quad f \quad {}_{A^p} < \infty\}$$

$\mathbb{D}$. dA $\frac{1}{\pi}dxdy$ $\mathbb{D}$, z $x + iy$. $\leq p < \infty$, A^p ${}_{A^p}$. A^p

$$H^p \quad \{f \quad \mathbb{D} \quad (_{0<r<1}\int_0^{2\pi} |f(re^{i\theta})|^p \frac{d\theta}{\pi})^{\frac{1}{p}} \quad f \quad {}_{H^p} < \infty\}.$$

The second author's research was partially supported by the AWM-NSF Mentoring Travel Grant and by a University Research Council Grant at Seton Hall University. The third author gratefully acknowledges support from the National Science Foundation.

$$\{|\int \bar{w} f \, dA| \quad \|f\|_{A^p} \leq \ \},$$

$$\{\|f\|_{A^p} \quad f \in A^p, l_i(f) \quad c_i, i \quad , \ldots, n\},$$

Proof. (.)

f_k A^p $I_i(f_k)$ c_i

$\leq i \leq n$ k , , . . .

A^p $p >$

A^2.

p

f

β_j,

$k(., \beta_j)$

$$k(z, w) \quad /(\quad - \bar{w}z)^2.$$

f $f(z)$

$(\phi(z$

$$F'(\cdot) \int | (p_m^*(z))|^2 Re(\prod_{i=1}^{n}(z-\beta_i)\psi_{m-n}(z))dA(z) .$$

ψ_{m-n} $i\psi_{m-n}$

$$\int | (p_m^*(z))|^2 Re(\prod_{i=1}^{n}(z-\beta_i)i\psi_{m-n}(z))dA(z) ,$$

$$\int | (p_m^*(z))|^2 \prod_{i=1}^{n}(z-\beta_i)\psi_{m-n}(z)dA(z)$$

ψ_{m-n} $m-n$. □

Lemma 2.7. *For each* $m \geq n$, $e^{p_m^*} \in H^2$, *and he e* H^2 *norm are bo nded.*

Proof.

$$p_m^*(z) \quad L(z) + h(z)q_{m-n}(z),$$

$L(z)$ c_i β_i (i $, \ldots, n)$, $h(z)$ $\prod_{i=1}^{n}(z-\beta_i)$, q_{m-n} $m-n$.

$$\int |e^{p_m^*(e^{i\theta})}|^2 d\theta \quad i\int |e^{p_m^*(z)}|^2$$

q_{m-n-1} $m-n-$.

$$zh'(z)q_{m-n}(z) \quad z\{\sum_{k=1}^{n}\prod_{i=1,i\neq k}^{n}(z-\beta_i)\}\{q_{m-n}(\beta_k)+(z-\beta_k)q$$

3. Another approach to regularity

n $\beta_1, \ldots, \beta_n$ $\mathbb{D}$, $c_1, \ldots, c_n$

L

, g_s … f_s … $L(f/g_s)$ … $\mathbb{D}$.

M,

$$\int |f$$

Proof. (*)

$$|a_s(z) - z| \le [illegible]\,|h_s(z) - \frac{[illegible]}{[illegible] - z}| \le [illegible] \sum_{n=1}^{\infty} |b_{n,s} - [illegible]|\,|z|^n.$$

$$|b_{n,s} - [illegible]| \quad |\frac{nt_0}{nt_0} - [illegible]|.$$

$$\frac{(\quad x)/x - [illegible]}{x^2}$$

[illegible] x [illegible] N

$$|\frac{nt_0}{nt_0} - [illegible]| \le N(nt_0)^2 \le N' n^2 s^2$$

[illegible] s, [illegible] N' [illegible]

$$|a_s(z) - z| \le N'' s^2 B(z),$$

$$B(z) \quad \sum_{n=1}^{\infty} n^2 |z|^n,$$

4. A discussion of the conjectured form of extremal functions

f^*

$\lambda \quad (\phi(z))_{A^2} \quad \phi(\beta_i$

A^2,

fi

A^p H^p ff

H

S A^2 S $\beta_1, \beta_2, \ldots, \beta_n$.

f^* $\frac{\partial}{\partial z}(H \prod_{j=1}^{n}(z-\beta_j)^2 Sg)$

g, . .,

$$\int \bar{f}^* \frac{\partial}{\partial z}(H \prod_{j=1}^{n}(z-\beta_j)^2 Sg) dA.$$

$$\int \bar{f}^* H \prod_{j=1}^{n}(z-\beta_j)^2 Sg d\bar{z} \qquad \int \bar{F} H \prod_{j=1}^{n}(z-\beta_j)^2 g \frac{dz}{}$$

$K \subset \mathbb{T}$ ... f^* ... A^2 ... $\beta_1, \beta_2, \ldots, \beta_n$,

$$f^* \quad \sum_{j=1}^{n} \frac{a_j}{(\; - \bar{\beta}_j z)^2}$$

Corollary 4.2. *If* f^* *i cyclic in* A^2*, i m be a ra ional f nc ion of he form* ( ).

( ) ... S f^* ... $n -$ ...), H ... p / ... $\mathbb{D}$ ... S. ... f^* ... ( ) ... (ff), ... fi ... S, ... H^2 ... fi ... B_r ... r A^2

$$B_r \quad \{f \quad e^{\phi} \quad f_{A^2} \leq r\},$$

$$\phi(z)$$

ν, $\mathbb{C}^n$

$$\Lambda(\nu) \quad (S(\nu)(\beta_j))_{j=1}^{n}.$$

$$S(\nu)(z) \quad \frac{1}{\pi}\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z}\,d\nu(\theta)$$

ν. $\Lambda(\Sigma_r)$ $\mathbb{C}^n$ A_r

, $R(e^{i\theta})$ ($\mathbb{T}$)

$E \subset \mathbb{T}$, $d\nu$ $s d\theta$ s

E fi $\mathbb{T} - E$,

$+\infty$ ( ), ( ) ( ),

( ).

ff $R >$ f^* R n $n -$ $R'(z)$ $\mathbb{C}$.

$$\{\int |F'(z)|^2 dA \quad F(\cdot) \quad , F'(\cdot) \quad , F''(\cdot) \quad b, F'(z) \ / \quad \mathbb{D}\}.$$

f F' c b

$$\{\int |f|^2 dA \quad f \ / \quad \mathbb{D}, f(\cdot) \quad , f'(\cdot) \quad c\}.$$

c f

$$f(z) \quad C(z-A)e^{\mu_0 \frac{z+1}{z-1}},$$

$\mu_0 \geq$, C, A, μ_0 $|c| \leq$

$$f^* \quad + cz$$

$F^* \quad z + \frac{c}{2}z^2$ $\mathbb{D}$ $|c| >$

$$f^* \quad hS,$$

h S

$$\int |f^*|^2 z^{n+2} dA \quad , n \quad , , , \ldots .$$

$c >$ f^*

Proposition 5.1. *If he ing lar fac or* S *of* f^* *ha a ocia ed ing lar mea re* $d\mu$ *ha i a omic i h a ingle a om, hen*

$$f^*(z) \quad C(z - \quad - \mu_0)e^{\mu_0 \frac{z+1}{z-1}}$$

here C *and he eigh* μ_0 *are niq ely de ermined by he in erpola ing condi ion .*

Remark.

$R(e^{i\theta})$ ... $R \geq$ ... $\mathbb{T}$,

$$R(e^{i\theta}) \quad \frac{(e^{i\theta}-a)(\;\; - \bar{a}e^{i\theta})}{e^{i\theta}} \quad |e^{i\theta}-a|^2,$$

$|a| \leq$ ...

$$\{\int R(e^{i\theta})d\mu(\theta) \quad \mu \leq \;, \mu \perp d\theta\}$$

μ ...

$$\int |h|^2|S_\mu|^2 dA \leq$$

... h ... R ... S_μ

[BK] C. Bénéteau and B. Korenblum, *ffi* H^p , Complex Analysis and Dynamical Systems, Israel Mathematical Conference Proceedings 16

[illegible]

[Ry2] V.G. Ryabych, [illegible], Siberian Math. J., XXVIII (1986), 212–217. (Russian)

[Vu] D. Vukotić, [illegible], Expositiones Math., 14 (1996), 313–352.

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