Note that, given the inner product (1.1), multiplication by z and differentiation are adjoint operators (at least formally): that is, for any polynomials p and q,

$$\langle zp(z), q(z)\rangle = \langle p(z), q'(z)\rangle. \tag{1.4}$$

In fact, if we define an inner product as in (1.1) where $e^{-|z|^2}$ is replaced by any continuous, radial weight $\omega(z)$ and require multiplication by z and differentiation

Definition 2.2. *Given an entire function f of finite order ρ; the type σ of the function is defined to be*

$$\sigma := \limsup_{r \to \infty} \ln M_f(r) n$$

an entire function can be computed from its Taylor coefficients (see, for example, [14]). More specifically, an entire function $\sum_{n=0}^{\infty} a_n z^n$ has order

$$\rho = \limsup_{n\to\infty} \frac{n \ln n}{\ln \frac{1}{|a_n|}}$$

and type

$$\sigma = \frac{1}{e\rho} \limsup_{n\to\infty} n|a_n|^{\rho/n}.$$

Notice that since the function f in (2.2) is in the Fock space F^2, f has order less than or equal to 2. On the other hand,

$$\rho = \limsup_{n\to\infty} \frac{n\ln(n)}{\ln(n\sqrt{n!})} \ge \limsup_{n\to\infty} \frac{n\ln(n)}{\ln(n) + \frac{1}{2}n\ln(n)} = 2.$$

Therefore, f has order 2. In addition,

$$\sigma = \frac{1}{2e} \limsup_{n\to\infty} n\left|\frac{1}{n\sqrt{n!}}\right|^{\frac{2}{n}} = \frac{1}{2}.$$

Notice that multiplying an entire function by z does not change its order or type. In particular, $zf(z)$ is a function of order 2 and type $1/2$. However, $zf(z)$ is not in the Fock space, since

$$\|zf(z)\|_2^2 = \sum_{n=1}^{\infty} \frac{(n+1)!}{n^2\, n!} = \sum_{n=1}^{\infty} \frac{n+1}{n^2} = \infty.$$

This example simultaneously shows that the Fock space cannot be defined simply in terms of order and type and that multiplication by z is not well-defined on the Fock space. (In fact, multiplication by z on the Fock space is one of the interesting examples of unbounded subnormal operators; see the introduction in [12] and the given references therein.)

One could also consider a simpler example, namely,

$$f(z) = \frac{\sin((1/2)z^2)}{z^2},$$

which is clearly an entire function of order 2 and type $1/2$. When $|z| = r$ gets large, $|f(z)|^2$ grows like $\frac{e^{r^2}}{r^4}$, which implies $\int_0^{2\pi}\int_0^{\infty} |f(re^{i\theta})|^2 e^{-r^2} r\,dr\,d\theta < \infty$, so that $f \in F^2$. On the other hand, $zf(z)$ is not in F^2. Indeed, this follows easily from the fact that when $|z| = r$ gets large,

$$|zf(z)|^2 e^{-r^2} r \approx \frac{1}{r}.$$

The above discussion shows that, from a certain point of view, the interesting functions in the Fock space are the ones of order 2 and type $1/2$. In fact, these Fock space functions have infinitely many zeros, which we now prove.

Proposition 2.4. *A function in the Fock space of order* 2 *and type* $1/2$ *must have infinitely many zeros.*

Proof. By the Hadamard factorization theorem, an entire function f of order 2 and type $1/2$ with finitely many zeros has the form

$$f(z) = P(z)e^{az+(1/2)z^2},$$

where P is a polynomial, and a is a constant. Then

$$\int_{\mathbb{C}} |f(z)|^2 e^{-|z|^2}\, dA(z) = \frac{1}{\pi} \iint_{\mathbb{R}^2} |P(z)|^2 e^{2\mathrm{Re}(a)x} e^{-\left(2\,\mathrm{Im}(a)y+y^2\right)}\, dx\, dy.$$

Since P is a polynomial, there are positive constants C and R such that $|P(z)| \geq C$ for $|z| \geq R.$ It follows that

$$\int_{\mathbb{C}} |f(z)|^2 e^{-|z|^2}\, dA(z) \geq \frac{C^2}{\pi} \left(\int_{\mathbb{R}\setminus[-R,R]} e^{2\mathrm{Re}(a)x}\, dx \right) \left(\int_{\mathbb{R}} e^{-\left(2\,\mathrm{Im}(a)y+y^2\right)}\, dy \right).$$

Clearly, the first integral in the right side of the above inequality diverges while the second integral converges. Therefore f cannot be in the Fock space. ∎

3. An Extremal Problem

Inspired by results in the theory of invariant subspaces of the Bergman space A^p which began with the seminal paper of Hedenmalm [10] for $p = 2,$ were extended by Duren, Khavinson, Shapiro, and Sundberg in [4, 5] for $p \neq 2,$ were developed by many others, and have now appeared in two books [6, 11], we investigate here the extremal function for a zero-based subspace of the Fock space F^2 associated with a finite zero set.

Note that in the Hardy and Bergman spaces, extremal functions come up naturally in connection with closed z-invariant subspaces, that is, closed subspaces M such that $zM \subset M.$ In the Fock space setting however, if M is a non-trivial closed subspace of F^2 and $f \in F^2$ satisfies $fM \subset M,$ then one can show that f must be a bounded function, and being entire, must thus be a constant. (See [9] for a more detailed discussion.) In particular, there are no closed z-invariant subspaces of the Fock space other than the trivial space. However, one can still examine the extremal functions.

Generally, if N is any closed subspace of $F^2,$ we consider the problem of maximizing $|f(0)|$ among all $f \in N$ with $\|f\|_2 = 1.$ (For convenience, we make the standing assumption that there is a function $f \in N$ with $f(0) \neq 0$; otherwise, we maximize a suitable derivative at the origin.) A function f which achieves this maximum is called an *extremal function* for $N.$ As in the Bergman space theory (see [11, Prop 3.5]), one can show that such an extremal function exists, belongs to $N,$ and is unique up to rotation by a unimodular constant. Henceforth, G_N will denote the unique extremal function for N with the property that $G_N(0) > 0.$

An extremal function G_N can be described in terms of the reproducing kernel for $N.$

Hilbert space with reproducing kernel $K(z,w) = e^{\bar{w}z}$. In other words, $K(\cdot,w)$ is a function in F^2 which satisfies $f(w) = \langle f, K(\cdot,w)\rangle$ for all $f \in F^2$ and all $w \in \mathbb{C}$. We remark that the existence (and uniqueness) of the kernel function follow from the Riesz representation theorem and the inequality (2.1), which shows that point evaluation is a bounded linear functional on F^2. The explicit formula for the kernel follows from the fact that the functions e_n in (1.2) form an orthonormal basis for F^2. (We omit the details.)

A closed subspace N of F^2 has its own reproducing kernel $K_N(z,w)$. Indeed, $K_N(\cdot,w)$ is the orthogonal projection of $K(\cdot,w)$ onto N, for each $w \in \mathbb{C}$. Using an argument similar to the one for the Bergman space, one can easily show that

$$G_N(z) = \frac{K_N(z,0)}{\sqrt{K_N(0,0)}}. \tag{3.1}$$

For our setting, we will only consider subspaces N of F^2 associated with a finite zero set. Indeed, if $A = \{a_1, a_2, \ldots, a_n\}$ is a finite set of distinct non-zero points in $\mathbb{C}$, let

$$N_A = \{f \in F^2 : f(a_j) = 0, \ 1 \le j \le n\}.$$

In what follows, for notational simplicity, the kernel function K_{N_A} and the extremal function G_{N_A} will be abbreviated K_A and G_A respectively.

Proposition 3.1. *Let $A = \{a_1, a_2, \ldots, a_n\}$ be a set of distinct non-zero points in $\mathbb{C}$. Then the extremal function G_A for the zero-based subspace N_A has infinitely many zeros.*

Proof. By considering a certain dual problem and using the reproducing property of kernels (see, e.g., [6, p. 14, 120]), it can be shown that G_A is a linear combination of the kernels $K(z,0) = 1$ and $K(z,a_k) = e^{\bar{a}_k z}$ for $k = 1, \ldots, n$. Thus, there are constants $c_0, \ldots, c_n$ such that

$$G_A(z) = c_0 + \sum_{k=1}^{n} c \tag{3.2}$$

Corollary 3.3. *If $a \neq 0$, then the single point extremal function is given by*

$$G_{\{a\}}(z) = \frac{1 - e^{\bar{a}(z-a)}}{\sqrt{1 - e^{-|a|^2}}}.$$

Proof. Combine (3.7) with (3.1). ∎

In the remainder of this section, we present some results about the zeros of these extremal functions. First of all, it follows easily from Corollary 3.3 that that the zeros of the single point extremal functions $G_{\{a\}}$ are given by $z_k = a + 2\pi i k/\bar{a}$, where k is an integer. For the two point extremal functions $G_{\{a,b\}}$, the situation is more complicated. We start with a basic lemma. Recall that $\mathbb{Q}$ denotes the set of rational numbers.

Lemma 3.4. *Suppose a, b are two distinct complex numbers such that $\frac{b}{a} = r$ belongs to $\mathbb{Q} \setminus \{0\}$, and let*

$$A = e^{|a|^2}, \quad \alpha = \frac{A^{r^2} - A^r}{A^r - A}, \quad \text{and} \quad \beta = \frac{A^{r^2+1} - A^{2r}}{A^r - A}. \tag{3.8}$$

(a) *If $r = \frac{q}{p}$, where $q > p > 0$ and p and q are relatively prime, then z is a solution to $G_{\{a,b\}}(z) = 0$ if and only if*

$$w^q - \alpha w^p + \beta = 0, \tag{3.9}$$

where $w = e^{\bar{a}z/p}$. Moreover, the polynomial equation (3.9) *has exactly q distinct solutions.*

(b) *If $r = -\frac{q}{p}$, where $q \geq p > 0$ and p and q are relatively prime, then z is a solution to $G_{\{a,b\}}(z) = 0$ if and only if*

$$w^{p+q} + \beta w^p - \alpha = 0, \tag{3.10}$$

where $w = e^{-\bar{a}z/p}$.

that any zero of (3.9) with multiplicity greater than one, if any, must be real, and that the real zeros of (3.9) are distinct.

So let $f(w) = w^q - \alpha w^p + \beta$ and assume that w_0 is a zero of f with multiplicity greater than one (clearly $w_0 \neq 0$). Thus, both f and f'

Theorem 3.5. *Suppose $a; b$ are distinct complex numbers such that* ^{b}

Then, by Lemma 3.4, $w_0 = e^{\overline{a} z_0/p}$ is a root of the equation $w^q - \alpha w^p + \beta = 0$, where α and β are as in (3.8). Notice that since $|\lambda| = 1$, the values of α and β in (3.8) remain unchanged if a and b are replaced with λa and λb. Since $w_0 = e^{\overline{\lambda a}(\lambda z_0)/p}$, it follows from Lemma 3.4 that λz_0 is a zero of $G_{\{\lambda a, \lambda b\}}$. (Notice that Lemma 3.4 can be applied in the case of the two points λa and λb, since $\lambda b/(\lambda a) = r = q/p \in \mathbb{Q} \setminus \{0\}$.) This proves the claim.

As a result of the claim, the number of distinct lines remains invariant if the zeros are rotated. Thus, from now on, we will assume that a and b are real.

Next, we obtain the upper bound on n. Observe that since α and β are real, the non-real roots of (3.9) must appear in conjugate pairs. We claim that if w_0 and $\overline{w_0}$ are conjugate roots of (3.9), then $K_\tau^{-1}(w_0)$ and $K_\tau^{-1}(\overline{w_0})$ are contained in the same line perpendicular to the real axis. (Recall that $\tau = a/p$.) Indeed, if z_0 satisfies $e^{z_0} = w_0$, then $e^{\overline{z_0}} = \overline{w_0}$. By (3.13), points in $K_\tau^{-1}(w_0)$ all have the same real part as z_0, while points in $K_\tau^{-1}(\overline{w_0})$ all have the same real part as Since $z_0) =$

the zeros of the extremal function. This implies in particular that there is no hope for an isometric or contractive divisor as we have in a Hardy or Bergman space situation (see also [23] for a discussion of this fact). Finally, by disturbing the original zeros slightly, we completely change the structure of the zeros of the extremal function, since the ratio of the two zeros can become real, or the zeros may no longer lie on the same line through the origin, for example. We do not know the exact structure of the zero set of the extremal function when the ratio of the zeros is irrational or non-real. In that case, the transcendental equation (3.11) can no longer be transformed into a polynomial equation. This type of equation has been studied in [13], and it may be possible to extract precise information in our setting from the results in that paper.

It should be mentioned that all upper bounds for n given by Theorem 3.5 can be achieved. For instance, taking $a = 1$ and $b = 4$; one can show that

Now, fix $A > 0$ and consider the extremal problem of finding

$$\inf\{\|f\|_2 : f(0) = 1, f'(0) = A, f \text{ non-vanishing in } \mathbb{C}\}. \tag{4.1}$$

By a standard normal family argument and the Cauchy-Schwarz inequality, it is easy to show that the extremal function f_* exists and is unique. (See, for example, [1] for the same argument in Bergman spaces.) The interpolating conditions immediately imply that $c = 0$ and therefore f_* has the form $f_*(z) = e^{az^2+Az}$, where $|a| < 1/2$.

Lemma 4.1. *The Maclaurin series for the extremal function f_* has real coefficients. Consequently, the constant a in the factorization of f_* must be real.*

Proof. Notice that the function $\overline{f_*(\bar{z})}$ is entire, has the same norm as f_*

Putting both calculations together gives

$$\|f\|_2^2 = \frac{1}{\pi}\frac{\sqrt{\pi}}{\sqrt{2a+1}}\exp\left(\frac{A^2}{1-2a}\right)\frac{\sqrt{\pi}}{\sqrt{1-2a}} = \frac{\exp\left(A^2/(1-2a)\right)}{\sqrt{1-4a^2}}$$

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